COLLEGE OF COMPUTER, MATHEMATICAL, & NATURAL SCIENCES

Limitations of Quantum Algorithms for Fluid Dynamics

Abtin Ameri [1] Joseph Carolan [2] Andrew Childs [2] Hari Krovi [3]

[1]Massachusetts Institute of Technology [2]University of Maryland [3]Riverlane

Joint Center for Quantum Information and Computer Science

Abstract

- Quantum algorithms have the potential to enable new applications in scientific computing
- A current candidate is quickly solving nonlinear differential equations
- To find applications of quantum, we need to know where not to look
- We provide formal evidence that (certain) quantum algorithms cannot efficiently simulate the Navier Stokes Equation, with infinite Reynolds number

Algorithm model

- Suppose we want to simulate a system described by equations Eqns, to time T
- Example: Navier Stokes Equations (incompressible)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$

- Let us ignore discretization, consider only scaling w.r.t. T
- Let $\mathcal{X}$ be the set of dynamical variables (velocities + pressures at all points for incompressible NSE)
- Let $\alpha_x(t)$ for $x \in \mathcal{X}$ be the value of x at time t
- We encode these variables quantumly as[a]

$$|\psi(t)\rangle \propto \sum_{x \in \mathcal{X}} \alpha_x(t) |x\rangle$$

- We allow the algorithm to make quantum queries to coefficients in Eqns

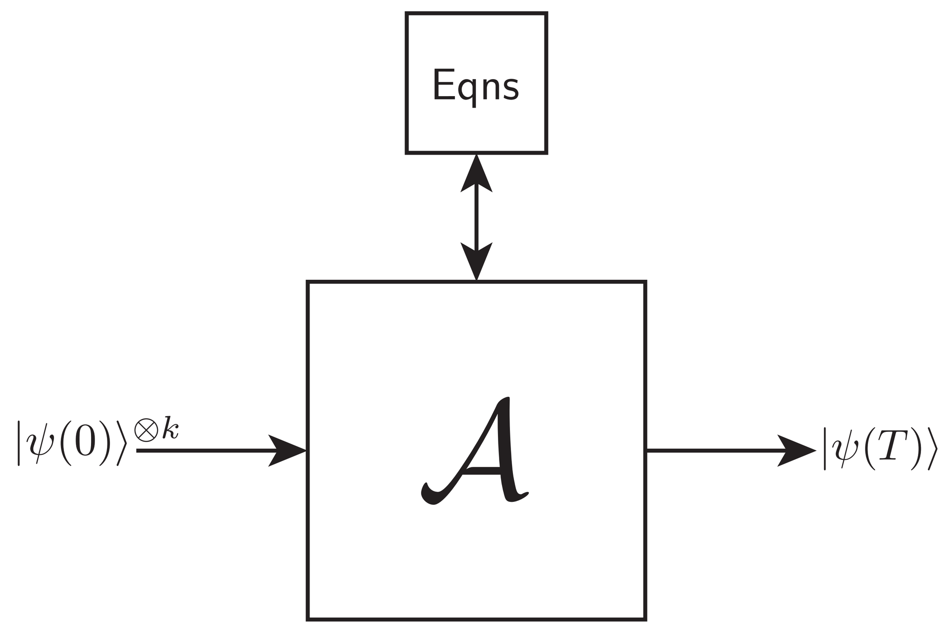

Figure 1: A general quantum algorithm for simulating nonlinear differential equations

- This encompasses nearly all known quantum algorithms
- Important caveat: it does not encompass *all* quantum algorithms
- Unconditional hardness is often beyond the abilities of modern computer science, not specific to this problem

How to prove lower bounds

- One way is to *reduce* the problem to one known to be hard
- A problem known to be hard: distinguishing close quantum states

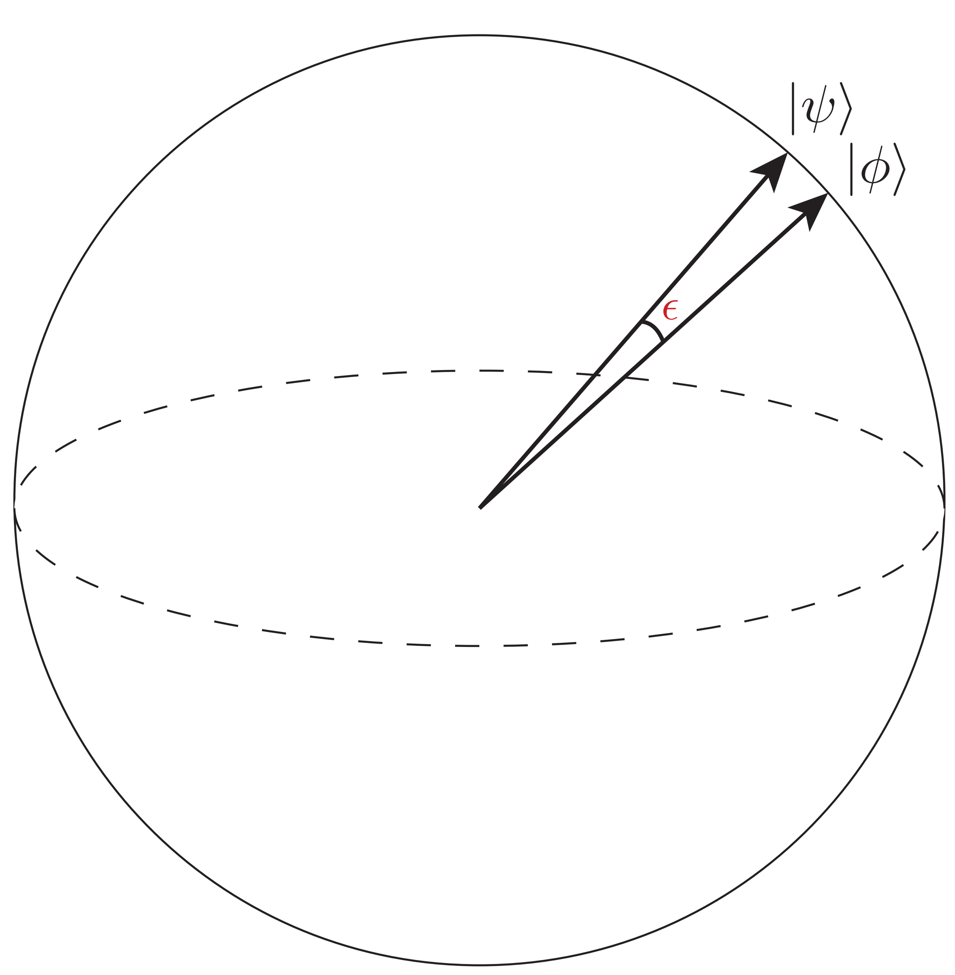

Figure 2: Given either $|\psi\rangle$ or $|\phi\rangle$, it is hard to tell which

- In particular, states ϵ-far require $\approx \epsilon^{-1}$ copies to distinguish
- When ϵ is exponentially small, this problem is hard
- However, non-linear dynamics can quickly pry apart close states
- This idea was introduced by Abrams & Lloyd (1998)

[a]we also need $\||\psi(0)\rangle\|$ as input.

Fluid instabilities

- We can prove a lower bound by finding exponentially diverging solutions
- Many known examples for the Navier Stokes Equations! Instabilities, chaos, etc.
- For this work, we use the Kelvin Helmholtz instability

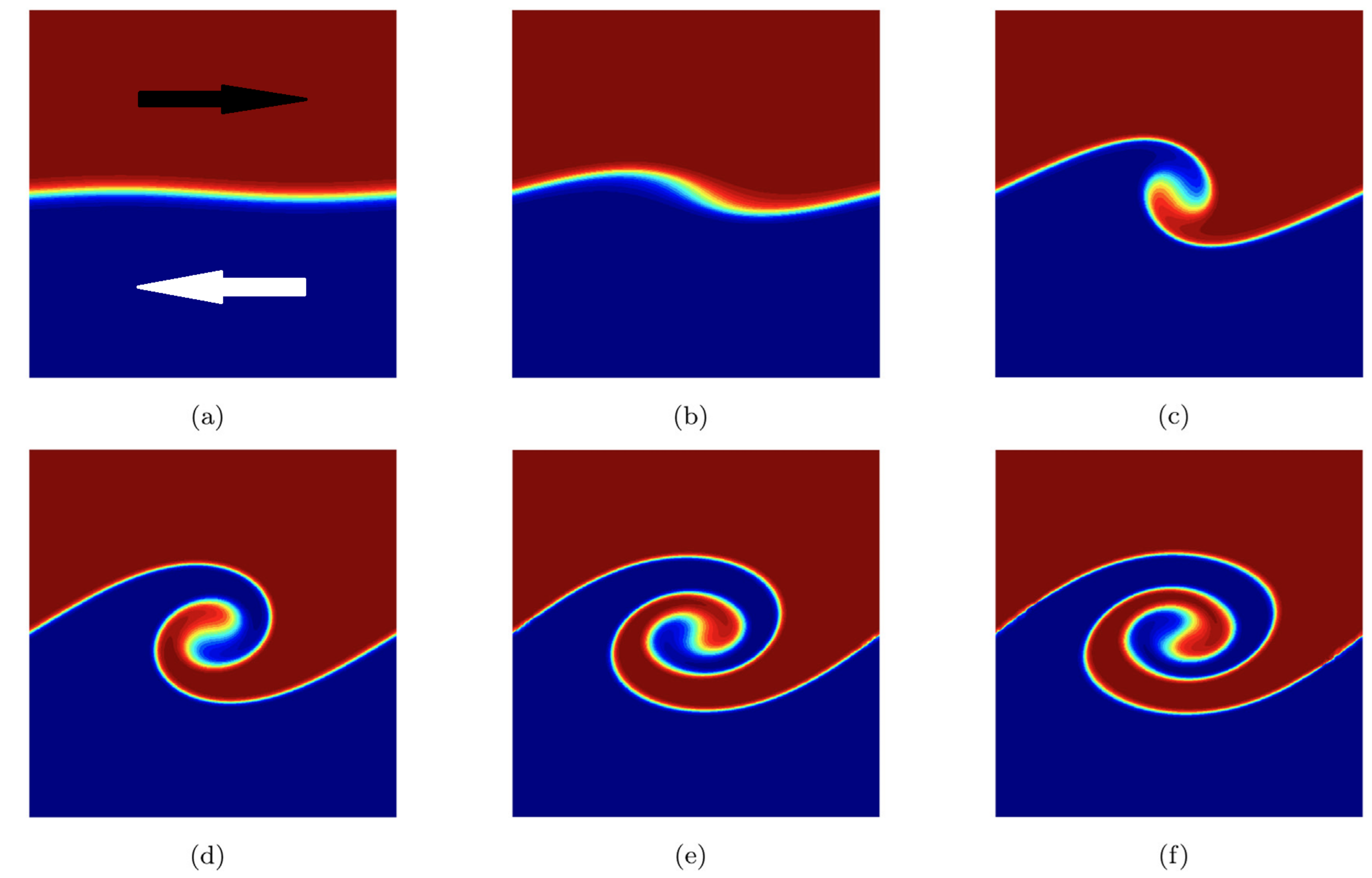

Figure 3: The Kelvin Helmholtz instability, Shah et al. (2023)

- Two sheets of fluid with a velocity discontinuity at the interface is a fixed point of NSE
- However, small perturbations to this fixed point result in exponential divergence, followed by roll-up
- We can use the initial exponential divergence to show a lower bound

The reduction

- Suppose that quantum algorithm $\mathcal{A}$ was able to simulate the NSE efficiently
- By *efficient*, we mean simulating to time T requires poly(T) resources
- Let $|\psi(0)\rangle$ be the fixed point in Kelvin-Helmholtz
- Let $|\phi(0)\rangle$ be the fixed point, perturbed by ϵ

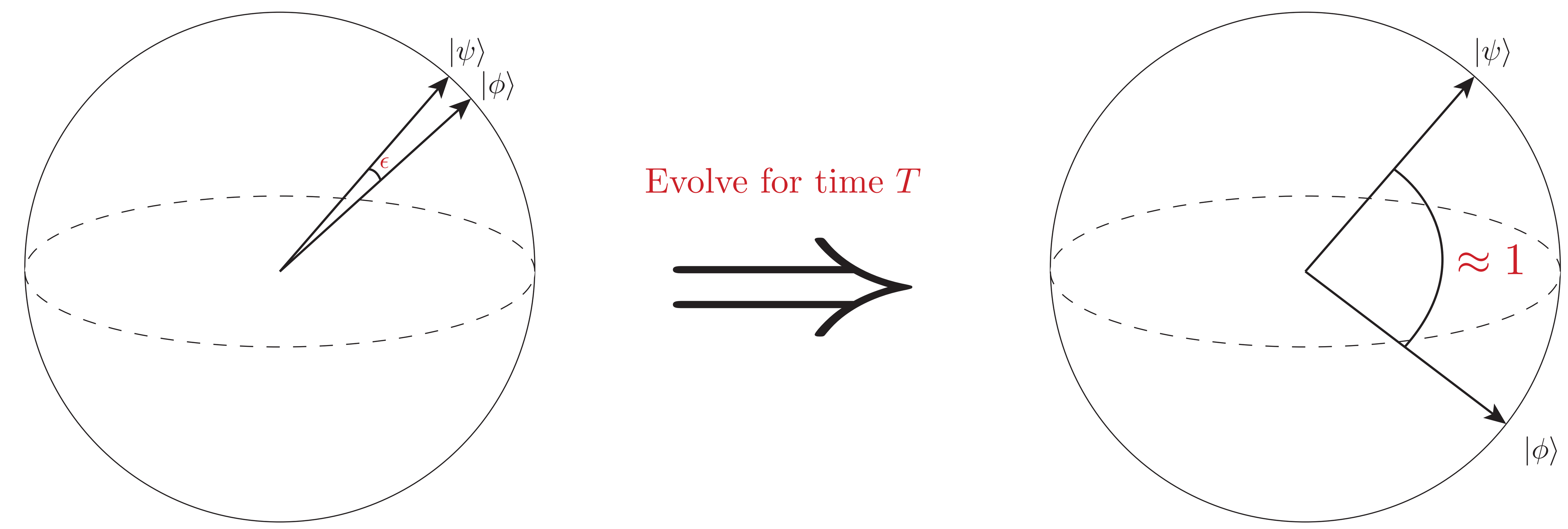

Figure 4: Evolution of states $|\psi\rangle$ and $|\phi\rangle$.

- After $T \approx \log \epsilon^{-1}$, the two states are far apart!
- So if $\mathcal{A}$ can simulate using poly T copies, it can also discriminate ϵ-far states with polylog(ϵ^{-1}) copies
- This is a contradiction! So $\mathcal{A}$ needs exp(T) copies, and therefore exp(T) resources

Omitted Details

- Divergence of trajectories needs to happen on a region which is a non-negligible fraction of the space
- Kelvin-Helmholtz is usually formulated with infinite spatial extent, we analyze a version on a compact spatial region with periodic boundary conditions
- One further needs to bound the difference between the linearized and full NSE, to show that the full dynamics also distinguish close states (This is W.I.P.)

References

Abrams, D. S., & Lloyd, S. (1998, Nov). Nonlinear quantum mechanics implies polynomial-time solution for *NP*-complete and # *P* problems. *Phys. Rev. Lett.*, *81*, 3992–3995.

Liu, J.-P., Øie Kolden, H., Krovi, H. K., Loureiro, N. F., Trivisa, K., & Childs, A. M. (2021). Efficient quantum algorithm for dissipative nonlinear differential equations. *Proceedings of the National Academy of Sciences*, *118*(35).

Shah, A., Sohaib, M., & Yuan, L. (2023). A numerical method for two-phase flow with its application to the kelvin–helmholtz instability problem. *Communications in Nonlinear Science and Numerical Simulation*, *125*, 107334.

Contact: jcarolan@umd.edu